ERGODIC AUTOMORPHISMS OF T^n ARE BERNOULLI SHIFTS

BY

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ABSTRACT

An automorphism of the n-dimensional torus T^n, none of whose eigenvalues is a root of unity includes on the canonical measure space of T^n a measure preserving transformation which is isomorphic to a Bernoulli shift.

The purpose of this paper is to prove its title. A measure preserving transformation S on a measure space (X,Ω,μ) is a Bernoulli shift if there exists a finite measurable partition $\mathscr{P} = \{P_1,\cdots,P_k\}$ on X such that the partitions $\{S^j\mathscr{P}\}_{j=-\infty}^{\infty}$ are independent and their join span Ω; such a partition is sometimes called a Bernoulli generator. Donald Ornstein proved [2] that the existence of partitions satisfying weaker conditions than independence of $\{S^j\mathscr{P}\}$, (such as "weak Bernoulli" (W.B.) or "very weak Bernoulli" (V.W.B.)), is sufficient to insure the existence of a Bernoulli generator for the σ-field they span under S. It also follows from Ornstein's work [3] that if $\{\mathscr{P}_k\}_{k=1}^{\infty}$ is a sequence of partitions such that $\mathscr{P}_{k+1}$ is a refinement of $\mathscr{P}_k$ and $\bigvee \mathscr{P}_k$ spans Ω under S, and if each $\mathscr{P}_k$ is V.W.B., then S is a Bernoulli shift on (X,Ω,μ). We show that in the case of an ergodic (algebraic) automorphism S of a finite dimensional torus T^n, every "nice" (see definition 3 below) partition of T^n satisfies a condition (almost weak Bernoulli, see definition 2 below) which seems weaker than "weak Bernoulli" but is clearly stronger than "very weak Bernoulli" which, by the preceding remarks, is sufficient to imply that S is a Bernoulli shift on T^n.

In §2 we extend the results to cover the case of S being an *epimorphism,* i.e. a homomorphism of T^n *onto* itself (not necessarily one-to-one) and obtain that the "natural extension" (see [5]) of (T^n,S) is a Bernoulli shift.

* This research was supported in part by the National Science Foundation grant GP–18884 and by the European Research Office of the U.S. Army contract DAJA–37–70–C–0701.

Received February 17, 1971

Partial results towards proving that ergodic automorphisms (and epimorphisms) of the torus, and more generally of a compact abelian group, are Bernoulli shifts were obtained in [1], [4], [6]. Although we do not settle here the general case of a compact abelian group, it seems likely that the methods presented here could be pushed to give that general case.

In §3 we prove a simple arithmetic lemma which is needed in the first sections; it is included for the sake of (relative) completeness.

I owe most of my education in the subject to D. Ornstein and R. Adler who deserve much of the credit for the results presented here.

1. Let (X,Ω,μ) be a (probability) measure space. All the partitions discussed below are assumed to be Ω measurable.

DEFINITION 1. Two finite partitions $\mathscr{A}$ and $\mathscr{B}$ of (X,Ω,μ) are *ε-independent*, $\varepsilon > 0$, if

$$\sum_{\mathscr{A}\vee\mathscr{B}} \left|\mu(A\cap B) - \mu(A)\mu(B)\right| < \varepsilon.$$

We shall use the following criterion to check ε-independence.

LEMMA 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be finite partitions of (X,Ω,μ). Let $\varepsilon > 0$ and $E \subset X$ such that $\mu(E) \subset \varepsilon^2$. Assume that to each $A \in \mathscr{A}$ corresponds a non-negative measurable function ϕ_A on X, and to each $B \in \mathscr{B}$ a function ψ_B such that*

$$\phi_A(t) \geqq 1 \text{ on } A\setminus E, \quad \psi_B(t) \geqq 1 \text{ on } B\setminus E \tag{1}$$

$$\sum_{A\in\mathscr{A}} \int \phi_A d\mu < 1+\varepsilon^2, \quad \sum_{B\in\mathscr{B}} \int \psi_B d\mu < 1+\varepsilon^2 \tag{2}$$

$$\int \phi_w \psi_B d\mu = \int \phi_A d\mu \int \psi_B d\mu. \tag{3}$$

Then $\mathscr{A}$ and $\mathscr{B}$ are $11\,\varepsilon$-independent.

PROOF. Denote

$$\mathscr{A}_1 = \{A; A\in\mathscr{A}, \int \phi_A d\mu \leqq (1+\varepsilon)\mu(A)\}$$

then

$$\sum_{A\notin\mathscr{A}_1}\int\phi_A d\mu \geqq (1+\varepsilon)\sum_{A\notin\mathscr{A}_1}\mu(A)$$

$$\sum_{A\in\mathscr{A}_1}\int\phi_A d\mu \geqq \sum_{A\in\mathscr{A}_1}\mu(A\setminus E) \geqq \sum_{A\in\mathscr{A}_1}\mu(A)-\varepsilon$$

hence, by (2)

$$1+\varepsilon^2 \geqq 1+\varepsilon\sum_{A\notin\mathscr{A}_1}\mu(A)-\varepsilon$$

or

$$\sum_{A\notin\mathscr{A}_1}\mu(A)\leqq 2\varepsilon. \tag{4}$$

Write

$$\mathscr{A}_0=\{A;A\in\mathscr{A},\int\phi_A d\mu\geqq(1-\varepsilon)\mu(A)\}$$

and notice that if $A\notin\mathscr{A}_0$, $\mu(A\cap E)\geqq\varepsilon\,\mu(A)$ so that $\varepsilon^2\geqq\mu(E)\geqq\sum_{A\notin\mathscr{A}_0}\mu(A\cap E)$ $\geqq\varepsilon\sum_{A\notin\mathscr{A}_0}\mu(A)$ or

$$\sum_{A\notin\mathscr{A}_0}\mu(A)\leqq\varepsilon. \tag{5}$$

For $A\in\tilde{\mathscr{A}}=\mathscr{A}_0\cap\mathscr{A}_1$ we have

$$(1-\varepsilon)\mu(A)\leqq\int\phi_A d\mu\leqq(1+\varepsilon)\mu(A) \tag{6}$$

and by (4) and (5)

$$\sum_{A\notin\tilde{\mathscr{A}}}\mu(A)\leqq 3\varepsilon. \tag{7}$$

Similarly, defining

$$\tilde{\mathscr{B}}=\{B;B\in\mathscr{B},(1-\varepsilon)\mu(B)\leqq\int\psi_B d\mu\leqq(1+\varepsilon)\mu(B)\}$$

we obtain

$$\sum_{B\notin\tilde{\mathscr{B}}}\mu(B)<3\varepsilon. \tag{8}$$

Finally, noticing that by (1), (2) and (3)

$$\phi_A\psi_B\geqq 1\quad\text{on }(A\cap B)\setminus E \tag{1'}$$

$$\sum\int\phi_A\psi_B<(1+\varepsilon^2)^2<1+3\varepsilon^2 \tag{2'}$$

and writing

$$\tilde{\mathscr{D}} = \{A \cap B;\, A \cap B \in \mathscr{A} \vee \mathscr{B},\ (1-\varepsilon)\mu(A \cap B) \leqq \int \phi_A \psi_B d\mu \leqq (1+\varepsilon)\mu(A \cap B)\}$$

we obtain as above

$$\sum_{A \cup B \notin \tilde{\mathscr{A}}} \mu(A \cap B) < 5\varepsilon \tag{9}$$

and the lemma follows from (7), (8), (9) and the observation that if $A \in \tilde{\mathscr{A}}$, $B \in \tilde{\mathscr{B}}$, $A \cap B \in \tilde{\mathscr{D}}$ then

$$\mu(A \cap B) \leqq (1-\varepsilon)^{-1} \int \phi_A \psi_B d\mu = (1-\varepsilon)^{-1} \int \phi_A d\mu \int \psi_B d\mu \leqq$$
$$\leqq (1-\varepsilon)^{-1}(1+\varepsilon)^2 \mu(A)\mu(B)$$

and

$$\mu(A \cap B) \geqq (1+\varepsilon)^{-1} \int \phi_A \psi_B d\mu = (1+\varepsilon)^{-1} \int \phi_A d\mu \int \psi_B d\mu$$
$$\geqq (1+\varepsilon)^{-1}(1-\varepsilon)^2 \mu(A)\mu(B). \qquad \blacktriangleleft$$

Let S be a measure preserving transformation on (X, Ω, μ).

DEFINITION 2. A (finite) partition $\mathscr{P}$ of X is *almost weak Bernoulli* (A.W.B.) for S if for all $\varepsilon > 0$ there exists an integer $K_0 = K_0(\varepsilon)$ such that for all $K > K_0$ and for all integers $N > K$, the partitions

$$\bigvee_{-N}^{-K} S^{-j}\mathscr{P} \quad \text{and} \quad \bigvee_{K}^{K^2} S^{-j}\mathscr{P}$$

are ε-independent.

It is clear that if $\mathscr{P}$ is A.W.B. then it is "very weak Bernoulli" (see [2] p. 182 for the definition of V.W.B.).

We consider now the case $X = T^n$, the n-dimensional torus, with the Lebesgue (Haar) σ-field and measure.

DEFINITION 3. A finite partition $\mathscr{P} = \{P_1, \cdots, P_r\}$ of T^n is "*nice*" if there exists a constant $l > 0$ such that for every positive integer m there exists a set E_m on T^n, $\mu(E_m) < m^{-2}$, and, for every $P \in \mathscr{P}$ there exists a non-negative trigonometric polynomial $f = f_{m.P}$ of degree (in each variable) less that m^l, such that $\sup \sum_{P \in \mathscr{P}} f_{m\,P}(t) \leqq 1 + m^{-2}$, and

$$f_{m.P}(t) \geqq 1 \text{ on } P \setminus E_m,\ f_{m.P}(t) < m^{-2} \text{ on } T^n \setminus (P \cup E_m).$$

REMARK. If each $P \in \mathscr{P}$ is a box, i.e. a product of intervals, then $\mathscr{P}$ is "nice". In fact, we can take $f_{m,P}$ to be the Fejér sum of order m^{10} (i.e. $l = 10$) of the characteristic function of P, multiplied by $1 + m^{-2}$.

The main result of this paper is

THEOREM 1. *Let S be an ergodic automorphism of T^n, then every "nice" partition of T^n is* A.W.B. *for S.*

COROLLARY *S is a Bernoulli shift on T^n.*

PROOF OF THEOREM 1. A typical atom in $\mathscr{A} = \bigvee_K^{K^2} S^{-m}\mathscr{P}$ has the form $A = \bigcap_{m=K}^{K^2} S^{-m}P_{j_m}$ $(j_m = 1,\cdots,r)$. We put $\phi_A = \prod_{m=K}^{K^2} S^{-m} f_{m,Pj_m}$, (where $(S^{-m}F)(t) = F(S^m t))$ and claim that $\phi_A(t) \geqq 0$ on T^n, and

(11)
$$\phi_A(t) \geqq 1 \quad \text{on } A \setminus \bigcup_K^{K^2} S^{-m}E_m,$$

$$\sum_{A \in \mathscr{A}} \int \phi_A d\mu = \int \sum_{A \in \mathscr{A}} \phi_A d\mu = \int \sum_K^{K^2} S^{-m}\Big(\sum_{P \in \mathscr{P}} f_{m,P}\Big) \leqq \prod_K^{K^2} (1 + m^{-2}).$$

Similarly, consider $\mathscr{B} = \bigvee_K^N S^m\mathscr{P}$ and for $B = \bigcap_{m=K}^N S^m P_{j_m} \in \mathscr{B}$ write $\psi_B = \prod_{m=K}^N S^m f_{m,P_{j_m}}$ and notice $\psi_B(t) \geqq 0$ and

(12)
$$\psi_B(t) \geqq 1 \quad \text{on} \quad B \setminus \bigcup_K^N S^m E_m$$

$$\sum_{B \in \mathscr{B}} \int \psi_B d\mu = \int \sum_{P \in \mathscr{A}} \psi_B d\mu = \int \prod_K^N S^m \Big(\sum_{P \in \mathscr{P}} f_{m,P}\Big) \leqq \prod_K^N (1 + m^{-2}).$$

Thus, given $\varepsilon > 0$ we can choose K_0' large enough to imply conditions (1) and (2) of Lemma 1 for all $K > K_0'$. We shall now show that there exists a constant K_0'' such that $K > K_0''$ implies (3). Writing $K_0 = \max(K_0', K_0'')$ Theorem 1 will then follow from Lemma 1.

We have not used so far the fact that S was an ergodic automorphism of T^n. This assumption is equivalent to saying that S maps exponentials to exponentials, the orbit of every non-trivial exponential being infinite. More specifically, S is given by an $n \times n$ matrix (which we again debote by S) with integral entries and determinant ± 1, and we have

$$Se^{i(\lambda \cdot t)} = e^{i(\lambda \cdot St)} = e^{i(S\lambda \cdot t)} \; \lambda \in Z^n.$$

If S had an eigenvalue which is a root of unity say, of order k, then $I - S^k$ would be singular and with integral coefficients and hence for some $\lambda \in Z^n$,

$\lambda = S^k\lambda$. Thus the assumption that S is ergodic implies that S has no eigenvalue which is a root of unity. We may clearly consider the matrix S as operating on R^n and we claim that if $V \subset R^n$ is an S-invariant subspace and the eigenvalues of $S|_V$ are all of modulus $\leqq 1$, then $V \cap Z^n = 0$. This follows from the fact that if all the eigenvalues of a non-singular matrix with integral entries (here $S|_{V\cap Z^n}$) have modulus $\leqq 1$ then they are all roots of unity (and the fact that each eigenvalue of $S|_{V\cap Z^n}$ is also an eigenvalue for $S|_V$ and S).

We now write $R^n = V \oplus V_1$ where V and V_1 are the S-invariant subspaces of R^n corresponding to the eigenvalues of S of modulus $\leqq 1$ and > 1 respectively. Using, e.g., the Euclidean norm we obtain that for an appropriate constant C if $x \in R^n$, $x = v(x) + v_1(x)\; v(x) \in V$ and $v_1(x) \in V_1$, then

$$\|v(x)\| < C\|x\| \quad \text{and} \quad \|v_1(x)\| \leqq C\|x\|. \tag{13}$$

We clearly have

$$\|S^m | V\| \leqq Cm^n. \tag{14}$$

Also, for some $\rho > 1$ and $m > m_0$

$$\|S^{-m} | V_1\| < \rho^{-m} \tag{15}$$

and in particular if $v_1 \in V_1$, $m > m_0$ then

$$\|S^m v_1\| \geqq \rho^m \|v_1\|. \tag{16}$$

Finally, by Lemma 3, which we prove in §3, we have some $C_1 > 0$ such that

$$\|v_1(\lambda)\| \geqq C_1 \|\lambda\|^{-n} \tag{17}$$

for all $\lambda \in Z^n$. Going back to the functions ϕ_A and ψ_B introduced above we notice first they all are trigonometric polynomials. We shall show now that the only frequency which is common to some ϕ_A and some ψ_B is $\lambda = 0$, which, by Parseval's formula, implies (3).

Let $b \in Z^n$ be a frequency which appears in ψ_B for some $B \in \mathscr{B}$. Then $b = \Sigma_{-K}^{-N} S^m b_m$ where b_m is a frequency in the corresponding $f_{|m|}$ and hence $\|b_m\| \leqq n|m|^l$. Now $v_1(b) = \Sigma_{-K}^{-N} v_1(S^m b_m) = \Sigma_{-K}^{-N} S^m(v_1(b_m))$ and assuming $K > m_0$ we have by (13) and (15)

$$\|v_1(b)\| \leqq Cn \sum_{m=K}^{\infty} m^l \rho^{-m}. \tag{18}$$

On the other hand let $a \neq 0$ be a frequency appearing in ϕ_A for some $A \in \mathscr{A}$. Then, as before,

$$a = S^K \left(\sum_K^{K^2} S^{m-K} a_m \right) \quad \text{with} \quad \| a_m \| \leqq nm^l .$$

We have

$$v \left(\sum_K^{K^2} S^{m-K} a_m \right) = \sum S^{m-K} v(a_m)$$

and by (13) and (14)

$$0 \neq \left\| v \left(\sum_K^{K^2} S^{m-K} a_m \right) \right\| \leqq Cn \sum_K^{K^2} (m-K)^n m^l \leqq CnK^{2n+2l+2}$$

hence by (17)

$$\left\| v_1 \left(\sum_K^{K^2} S^{m-K} a_m \right) \right\| > C_2 K^{-2n(n+l+1)}$$

and by (16)

$$\text{(19)} \qquad \| v_1(a) \| = \left\| S^K v_1 \left(\sum_K^{K^2} S^{m-K} a_m \right) \right\| > C_2 \rho^K K^{-2n(n+l+1)} .$$

It is clear that for $K > K_0$, (18) and (19) are inconsistent, and the proof is complete.

2. The argument in the proof of Theorem 1 can be refined somewhat to enable us to deal with epimorphisms of T^n. If S is a non-singular $n \times n$ matrix with integral entries, S defines an epimorphism of T^n i.e., a (usually many to one) homomorhism of T^n *onto* itself. S is then measure preserving in the sense that for all measurable E on T^n we have $\mu(S^{-1}(E)) = \mu(E)$, (the pre-image of every point under S is a set of $|\det S|$ points, and $d\mu(St) = |\det S| d\mu(t)$) and it is known [5] that under such conditions, (T^n, S) admits a "natural extension" (analogous to the extension of the "one sided shift" to the "two sided shift"). Our Theorem 2 implies that this "natural extension" of (T^n, S) is Bernoulli.

THEOREM 2. *Let S be an ergodic epimorphism of T^n, let $\mathscr{P}$ be a "nice" partition of T^n and let $\varepsilon > 0$. Then there exists an integer K_0 such that for every $K > K_0$ and arbitrary $N > 0$, the partitions $\bigvee_0^{K^2} S^{-m}\mathscr{P}$ and $\bigvee_{K^2+K}^{K^2+K+N} S^{-m}\mathscr{P}$ are ε-independent.*

PROOF. We follow the lines of the proof of Theorem 1; the only additional information that we need is the following:

LEMMA 2. *Let S be as above. Let l be an integer. There exists a constant K_0 such that if $K > K_0$, N arbitrary, and f_m are trigonometric polynomials* *on T^n of degree (in each variable) bounded by K^{2l} for $m \leqq K^2$ and by m^l for $m \geqq K^2$, then the only frequency common to $\prod_1^{K^2} S^{-m}f_m$ and $\prod_{K^2+K}^{K^2+K+N} S^{-m}f_m$, is zero (where $(S^{-m}f)(t) = f(S^m t)$).*

PROOF. S (more precisely its adjoint) is a 1-1 endomorphism of Z^n which can be extended as before to an automorphism of R^n. We decompose $R^n = V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots \oplus V_r$ such that V_j, $j = -1, 0, \cdots, r$ are S invariant, all the eigenvalues of $S|_{V_{-1}}$ have modulus less than one, all the eigenvalues of $S|_{V_0}$ have modulus one, and for $j = 1, \cdots, r$, all the eigenvalues of $S|_{V_j}$ have the same modulus ρ_j, where $\rho_0 = 1 < \rho_1 < \cdots < \rho_r$. We denote by ρ_{-1} the maximum modulus of the eigenvalues of $S|_{V_{-1}}$. This decomposition is given by the "Jordan canonical form" theorem, which also shows that if $v \in V_j$ and $m > 0$ is an integer, then

$$\text{const } m^{-n}\rho_j^m \| v \| \leqq \| S^m v \| \leqq \text{const } m^n \rho_j^m \| v \| . \tag{20}$$

For any $\lambda \in Z^n$ we write $\lambda = \sum_{-1}^{r} v_j(\lambda)$ where $v_j(\lambda) \in V_j$. We also notice that the assumption of ergodicity implies $(V_{-1} \oplus V_0) \cap Z^n = 0$.

Let λ be a frequency which appears (with non-zero coefficient) in $\prod_1^{K^2} S^{-m}f_m$. Then $\lambda = \sum_{m=1}^{K^2} S^m \lambda_m$ where

$$\| \lambda_m \| \leqq \text{const}\, K^{2l} \tag{21}$$

for all m. We clearly have

$$v_j(\lambda) = \sum_{m=1}^{K^2} S^m(v_j(\lambda_m)) \tag{22}$$

which, by (20) and (21) implies for $j \geqq 0$

$$\| v_j(\lambda) \| \leqq \text{const} \sum_{m=1}^{K^2} m^n \rho_j^m K^{2l} \leqq \text{const}\, K^{2n+2l+2} \rho_j^{K^2} \tag{23}$$

while

$$\| v_{-1}(\lambda) \| \leqq \text{const}\, K^{2n+2l+2}. \tag{24}$$

On the other hand, if λ' is a frequency which appears in $\prod_{K^2+K}^{K^2+K+N} S^{-m}f_m$ then $\lambda' = S^{K^2+K}\lambda''$ where $\lambda'' = \sum_{m=0}^{N} S^m \lambda''_{m+K^2+K}$. We have

$$\| v_{-1}(\lambda'') \| \leqq \text{const} \sum_{m=0}^{N} m^n \rho_{-1}^m (m + K^2 + K)^l \leqq \text{const}\, K^{2l} \tag{25}$$

and if λ' is also a frequency of $\prod_1^{K^2} S^m f_m$ then by (23) and (20)

$$\| v_j(\lambda'') \| \leqq \text{const}\, K^{2n} \rho_j^{-K^2-K} K^{2n+2l+2} \rho_j^{K^2} = \text{const}\, K^{l_1} \rho_j^{-K} . \tag{26}$$

It follows that $\| v_{-1}(\lambda'') + v_0(\lambda'') \|$ is bounded by a power of K while $\| \sum_1^r v_j(\lambda'') \|$ decreases exponentially with K; by Lemma 3 this implies $\lambda'' = 0$ hence $\lambda' = 0$. This completes the proof of Lemma 2 and of Theorem 2.

3.

Definition. A subspace V of R^n is an *eigenspace* of an $n \times n$ matrix S if $R^n = V \oplus V'$ with V and V_1 both invariant under S and $S|_V$, $S|_{V_1}$ have no common eigenvalue.

It is clear that if V is an eigenspace of S with V_1 as above and if $\mathscr{P}$ is the minimal polynomial of $S|_V$, then $\mathscr{P}(S)$ is invertible on V_1.

Lemma 3. *Let S be an $n \times n$ matrix with integral coefficients and let V be an m-dimensional eigenspace of S. Assume $V \cap Z^n = 0$. Then there exists a constant C such that for every $\lambda \in Z^n$, $d(\lambda, V) \geqq C \| \lambda \|^{-m}$, where $d(\lambda, V)$ is the (Euclidean) distance of λ to V.*

Proof. Let $\mathscr{P}(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_0$ $(k \leqq m)$ be the minimal polynomial of $S|_V$; notice that the coefficients a_j are real numbers. There exists some $\eta > 0$ such that if $F(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_0$ with $|b_j - a_j| < \eta$ $j = 0 \cdots k-1$ then the null space of $F(S)$ is contained in V. This follows from the fact that, if η is small, $\mathscr{P}(S)F(S)$ is very close to $(\mathscr{P}(S))^2$ which is invertible on V_1, and its nullity is therefore bounded by m and hence its null space, which clearly contains that of $F(S)$, is exactly V. Thus, if η is sufficiently small, $F(S)\lambda \neq 0$ for $\lambda \in Z^n$, $\lambda \neq 0$.

By Dirichlet's theorem, for every positive integer Q, there exist integers $q, r_0, \cdots, r_{k-1}$, $q \leqq Q^k$, such that $|a_j - r_j/q| \leqq 1/qQ$ for $j = 0, \cdots, k-1$. Denote $\mathscr{P}_Q(x) = x^k + \sum_0^{k-1} q^{-1} r_j x^j$, then for $Q > Q_0$ $\mathscr{P}_Q(S)\lambda \neq 0$ for $\lambda \in Z^n$, $\lambda \neq 0$, and, since $q\mathscr{P}_Q(S)$ has integral coefficients, $\| \mathscr{P}_Q(S)\lambda \| > 1/q$ for any such λ.

For $\lambda \neq 0$, $\lambda \in Z^n$ let v be its projection on V and write

$$\mathscr{P}_Q(S)\lambda = \mathscr{P}_Q(S)(\lambda - v) + (\mathscr{P}_Q(S) - \mathscr{P}(S))v$$

hence, if $C > (\sum_0^{k-1} |a_j| + k) \| S \|^k$,

$$\frac{1}{q} \leqq \| \mathscr{P}_Q(S)\lambda \| \leqq C\left(d(\lambda, V) + \frac{\| \lambda \|}{qQ}\right)$$

and choosing $Q = 2C \| \lambda \|$ we get

$$d(\lambda, V) > \frac{1}{2Cq} \geqq \frac{1}{2CQ^k} > C_1 \| \lambda \|^{-k}$$

and the proof is complete.

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